Circularly Polarized Wave by Two Linearly Polarized Waves

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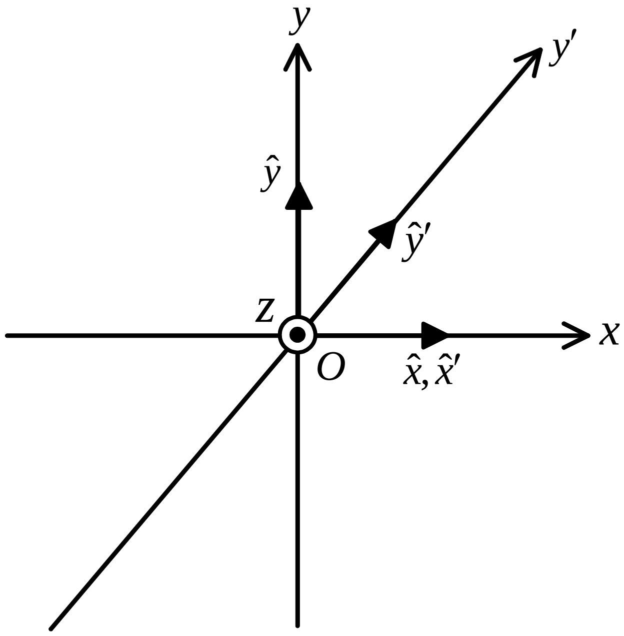

Fig. 1 Coordinates

The condition that the sum of two linearly polarized waves $\hat{x}' A \exp(-jkz), \hat{y}' B \exp(-jkz)$, traveling toward $+z$ direction as shown in Fig. 1, becomes circularly polarized wave is proved here, where k is the wavenumber.

The coordinates are shown in Fig. 1. In (x', y', z') coordinates, x' and y' axes are crossing with angle θ and x' and z' axes coincide with x and z axes, respectively. $\hat{x}, \hat{y}, \hat{x}'$ and $\hat{y}'$ indicate unit vectors of x, y, x' and y' axes, respectively. $\hat{x}'$ and $\hat{y}'$ are expressed by $\hat{x}$ and $\hat{y}$ as follows.

$$\begin{cases} \hat{x}' = \hat{x} \\ \hat{y}' = \hat{x}\cos\theta + \hat{y}\sin\theta \end{cases} \quad (1)$$

Consider the sum of two linearly polarized waves whose axes of polarization are x' and y', respectively, as shown in Fig. 1.

$$\hat{x}' A + \hat{y}' B \quad (2)$$

where A and B are complex numbers (Phasor representation) as follows

$$\begin{cases} A = R_A \exp(j\theta_A) \\ B = R_B \exp(j\theta_B) \end{cases} \tag{3}$$

where R_A, θ_A, R_B and θ_B are real numbers.

By substituting Eq.(3) into Eq.(2) and using Eq.(1)

$$\hat{x} R_A \exp(j\theta_A) + (\hat{x}\cos\theta + \hat{y}\sin\theta) R_B \exp(j\theta_B)$$

$$\hat{x}\{R_A \exp(j\theta_A) + R_B \exp(j\theta_B)\cos\theta\} + \hat{y} R_B \exp(j\theta_B)\sin\theta$$

The circular polarization condition in $x-y$ plane is applied here.

$$R_A \exp(j\theta_A) + R_B \cos\theta \exp(j\theta_B) = \pm j R_B \sin\theta \exp(j\theta_B)$$

where + sign (upper) in right hand side means the right hand circularly polarized wave (RHCP) while – sign (lower) means the left hand one (LHCP) when the wave propagates toward +z direction.

$$R_A \exp(j\theta_A) + R_B \cos\theta \exp(j\theta_B) \mp j R_B \sin\theta \exp(j\theta_B) = 0$$

$$R_A \cos\theta_A + R_B \cos\theta \cos\theta_B \pm R_B \sin\theta \sin\theta_B$$
$$+ j\{R_A \sin\theta_A + R_B \cos\theta \sin\theta_B \mp R_B \sin\theta \cos\theta_B\} = 0$$

$$\begin{cases} R_A \cos\theta_A + R_B(\cos\theta\cos\theta_B \pm \sin\theta\sin\theta_B) = 0 \\ R_A \sin\theta_A + R_B(\cos\theta\sin\theta_B \mp \sin\theta\cos\theta_B) = 0 \end{cases}$$

$$\begin{cases} R_A \cos\theta_A + R_B(\cos\theta\cos\theta_B \pm \sin\theta\sin\theta_B) = 0 \\ R_A \sin\theta_A \mp R_B(\sin\theta\cos\theta_B \mp \cos\theta\sin\theta_B) = 0 \end{cases}$$

$$\begin{cases} R_A \cos\theta_A + R_B \cos(\theta \mp \theta_B) = 0 \quad (4) \\ R_A \sin\theta_A \mp R_B \sin(\theta \mp \theta_B) = 0 \quad (5) \end{cases}$$

By Eq.(4)

$$R_B = -\frac{R_A \cos\theta_A}{\cos(\theta \mp \theta_B)} \tag{6}$$

Substituting this equation into Eq. (5) yields

$$R_A \sin\theta_A \pm R_A \cos\theta_A \tan(\theta \mp \theta_B) = 0$$

$$\sin\theta_A \pm \cos\theta_A \tan(\theta \mp \theta_B) = 0$$

$$\sin\theta_A = \mp\cos\theta_A \tan(\theta \mp \theta_B)$$

$$\mp\sin\theta_A = \cos\theta_A \tan(\theta \mp \theta_B)$$

$$\mp\tan\theta_A = \tan(\theta \mp \theta_B)$$

$$\tan(\mp\theta_A) = \tan(\theta \mp \theta_B)$$

$$\mp\theta_A + n\pi = \theta \mp \theta_B$$

where n is an arbitrary integer number.

$$\mp\theta_A - \theta + n\pi = \mp\theta_B$$

$$\theta_B = \mp(\mp\theta_A - \theta + n\pi)$$

$$\theta_B = \theta_A \pm \theta \mp n\pi$$

$$\theta_B = \theta_A \pm \theta + n'\pi$$

By substituting this into Eq.(6)

$$R_B = -\frac{R_A \cos\theta_A}{\cos(\mp\theta_A + n\pi)} = -\frac{R_A \cos\theta_A}{(-1)^n \cos(\mp\theta_A)} = -\frac{R_A \cos\theta_A}{(-1)^n \cos\theta_A} = (-1)^{n-1} R_A$$

The following two conditions are obtained for circular polarization

$$\begin{cases} R_B = (-1)^{n-1} R_A \\ \theta_B = \theta_A \pm \theta \mp n\pi \end{cases}$$

$$\begin{cases} R_B = R_A \\ \theta_B = \theta_A \pm \theta + m\pi \quad (m:\ \text{odd}) \end{cases}$$

where + sign (upper) in right hand side means the right hand circularly polarized wave (RHCP) while – sign (lower) means the left hand one (LHCP) when the wave propagates toward +z direction.