重積分の変数変換とヤコビアン

- 円の面積の計算
- 球の体積・面積の計算

平野拓一

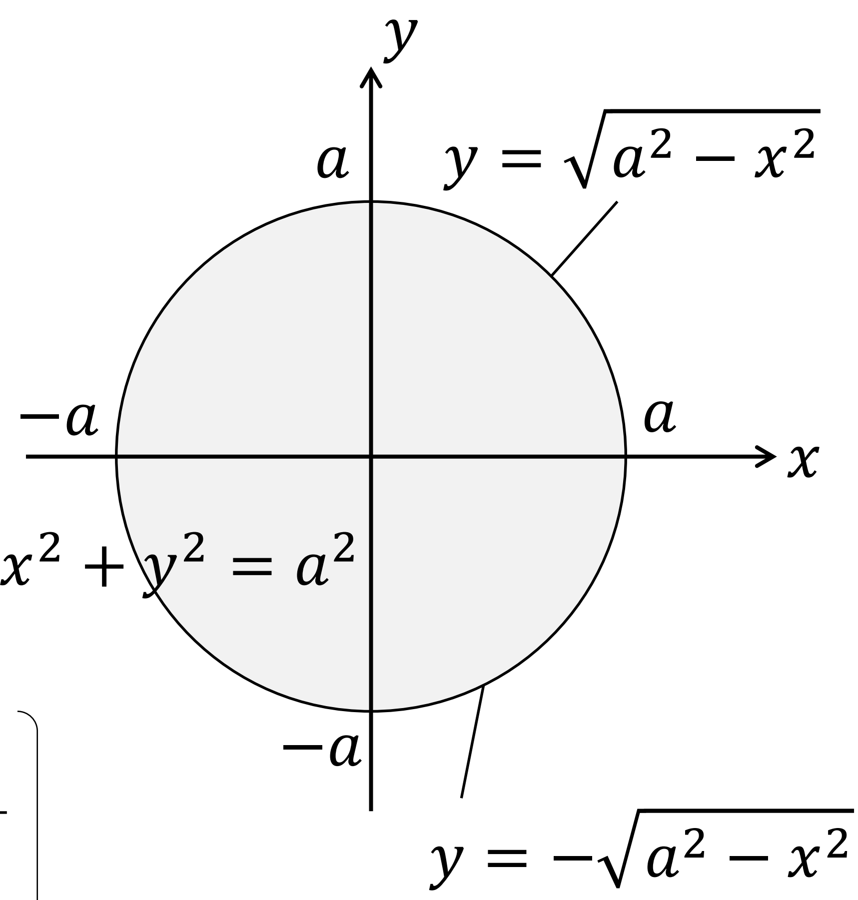
円内での積分（円の面積）

$$\iint_S dxdy = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dxdy$$

$$= \int_{x=-a}^{a}\left(\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\right)dx$$

$$= \int_{x=-a}^{a}[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$= 2\int_{x=-a}^{a}\sqrt{a^2-x^2}dx$$



$$x = a\sin\theta$$

$$\frac{dx}{d\theta} = a\cos\theta$$

x	$-a$	$\to$	a
θ	$-\pi/2$	$\to$	$\pi/2$

$$= 2\int_{\theta=-\pi/2}^{\pi/2} a\sqrt{1-\sin^2\theta}\ (a\cos\theta\, d\theta) = 2a^2\int_{\theta=-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$= 2a^2\int_{\theta=-\pi/2}^{\pi/2}\frac{1}{2}\{\cos(2\theta)+1\}d\theta = a^2\int_{\theta=-\pi/2}^{\pi/2}d\theta = a^2[\theta]_{-\pi/2}^{\pi/2} = \pi a^2$$

極座標に変換して積分

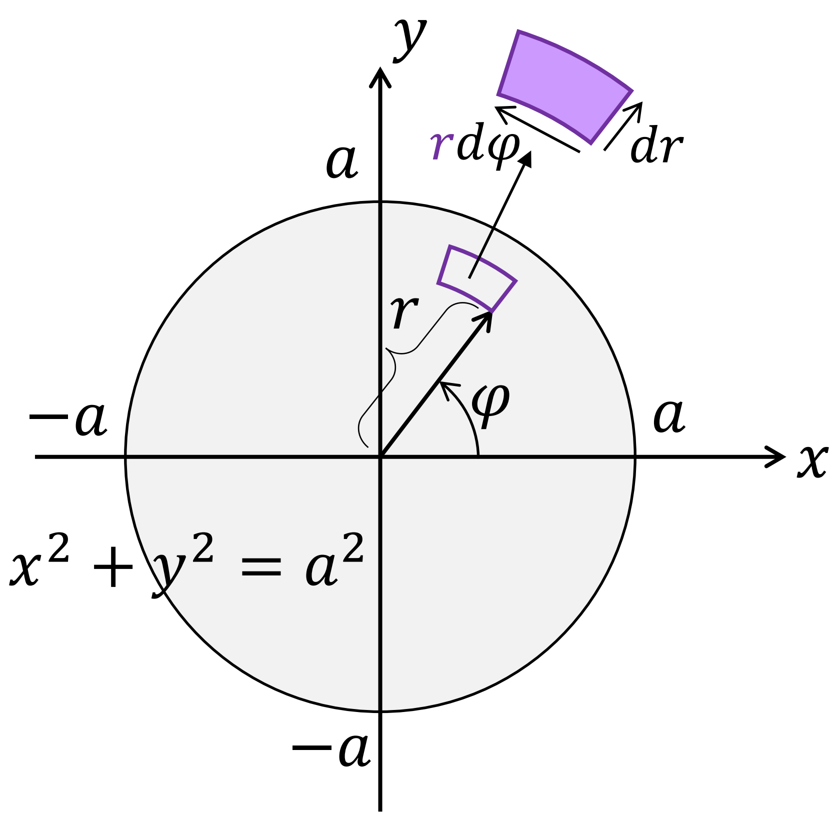
$$\iint_S dxdy = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dxdy$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\varphi, \dfrac{\partial x}{\partial \varphi} = -r\sin\varphi \\ \dfrac{\partial y}{\partial r} = \sin\varphi, \dfrac{\partial y}{\partial \varphi} = r\cos\varphi \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

$|J| = r$ （ヤコビアン; Jacobian）



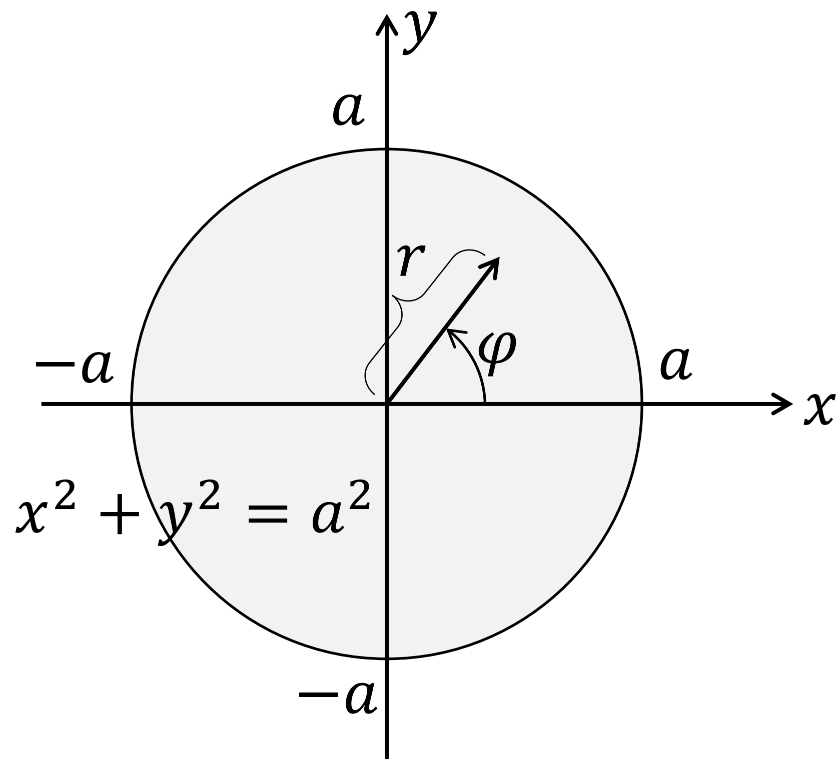
$$= \int_{r=0}^{a} \int_{\varphi=0}^{2\pi} |J| dr d\varphi = \int_{r=0}^{a} \int_{\varphi=0}^{2\pi} r dr d\varphi = \int_{r=0}^{a} r dr \int_{\varphi=0}^{2\pi} d\varphi$$

$$= \left[\frac{r^2}{2}\right]_0^a [\varphi]_0^{2\pi} = \pi a^2$$

ヤコビアンの意味

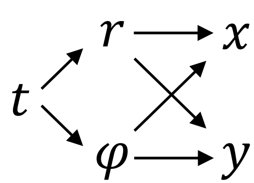
$$J=\frac{\partial(x,y)}{\partial(r,\varphi)}=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{bmatrix}=\begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix}$$

$|J|=r$ （ヤコビアン）



全微分

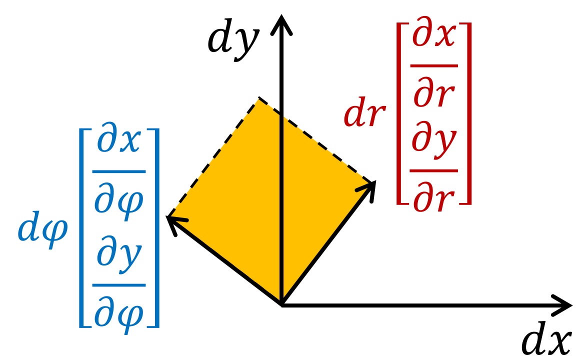
$$\begin{cases} dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial \varphi}d\varphi\\ dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial \varphi}d\varphi\end{cases}$$



$$\begin{bmatrix}dx\\ dy\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial \varphi}d\varphi\\ \frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial \varphi}d\varphi\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{bmatrix}\begin{bmatrix}dr\\ d\varphi\end{bmatrix}$$

$$=dr\begin{bmatrix}\frac{\partial x}{\partial r}\\ \frac{\partial y}{\partial r}\end{bmatrix}+d\varphi\begin{bmatrix}\frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial \varphi}\end{bmatrix}$$

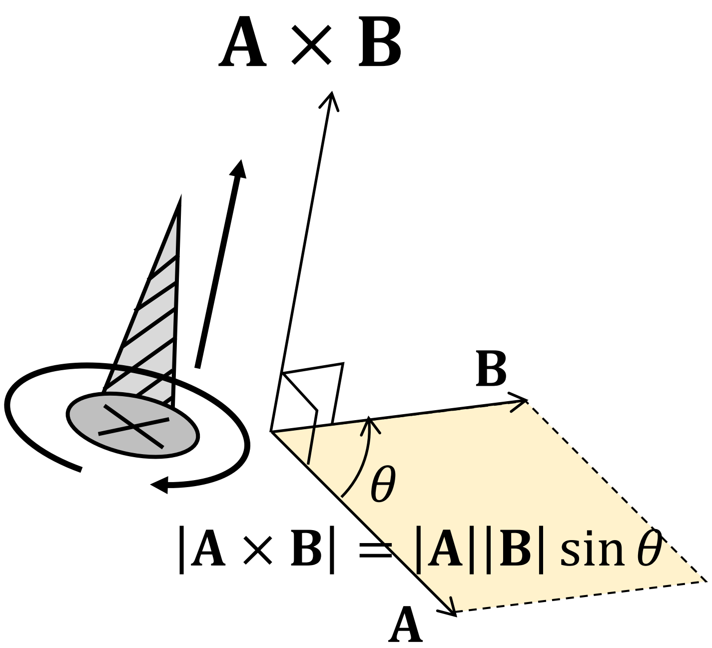
⇒ この２つのベクトルで生成される平行四辺形の面積



drにより動く$\begin{bmatrix}dx\\ dy\end{bmatrix}$の成分　　$d\varphi$により動く$\begin{bmatrix}dx\\ dy\end{bmatrix}$の成分

外積と平行四辺形の面積

$$
\begin{aligned}
&\mathbf{A}\times\mathbf{B}\\
&=\left(\hat{x}A_x+\hat{y}A_y+\hat{z}A_z\right)\times\left(\hat{x}B_x+\hat{y}B_y+\hat{z}B_z\right)\\
&=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\\
&\hat{x}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\hat{y}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\hat{z}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\\
&=\hat{x}\left(A_yB_z-A_zB_y\right)\\
&\ -\hat{y}\left(A_xB_z-A_zB_x\right)\\
&\ +\hat{z}\left(A_xB_y-A_yB_x\right)
\end{aligned}
$$



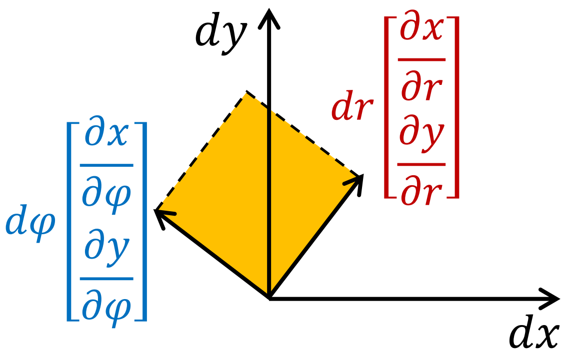
平行四辺形の面積

$$|\mathbf{A}\times\mathbf{B}|_z = A_xB_y - A_yB_x$$

向き：（$\theta < 180°$として$\mathbf{A}$から$\mathbf{B}$に回して）
ネジが閉まる方向
（右に回すとしまる）
大きさ：$\mathbf{A}, \mathbf{B}$のベクトルで形成される平行四辺形の面積

平行四辺形の面積としてヤコビアンを計算

$$\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \begin{vmatrix} dr\dfrac{\partial x}{\partial r} & dr\dfrac{\partial y}{\partial r} \\ d\varphi\dfrac{\partial x}{\partial \varphi} & d\varphi\dfrac{\partial y}{\partial \varphi} \end{vmatrix}$$



(行列式の公式: 行・列に同じ係数がかかっていると、外にスカラー倍として出せる。)

$$= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} dr d\varphi$$

(行列式の公式: $|A| = |A^t|$)

$$= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} dr d\varphi$$

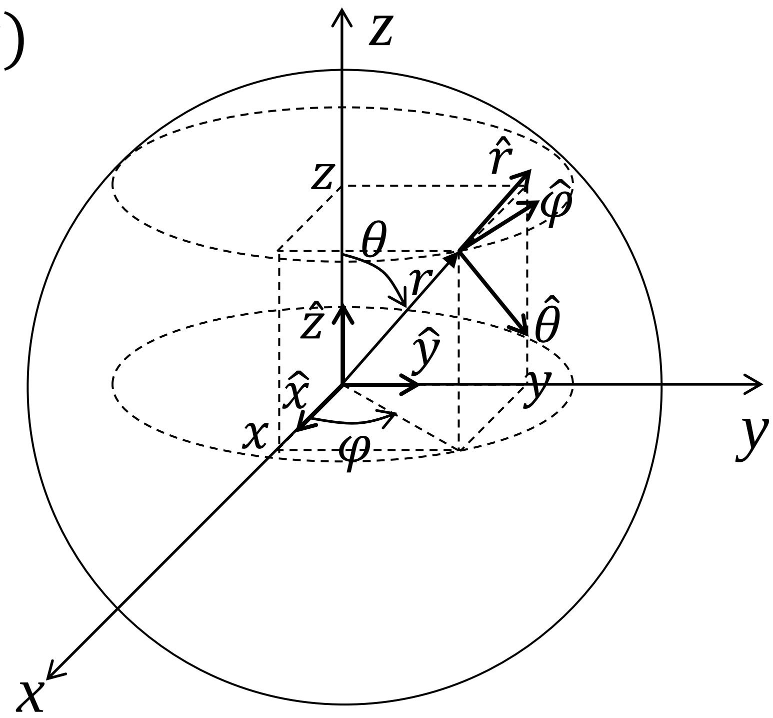
ヤコビアン(Jacobian)

球座標系(Spherical Coordinates)

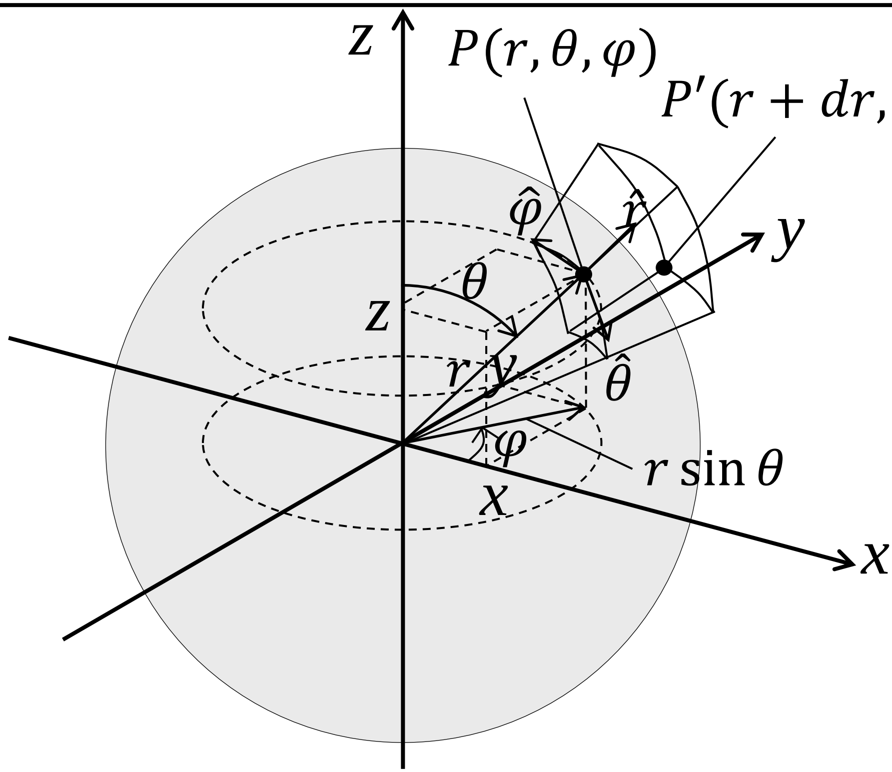
変換(Transform): $(x, y, z) \leftrightarrow (r, \theta, \varphi)$

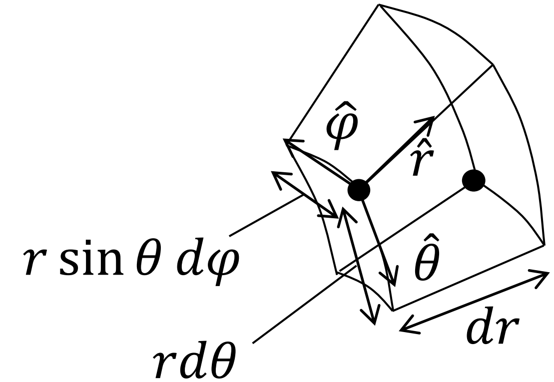
$$(r, \theta, \varphi) \to (x, y, z) \qquad \begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

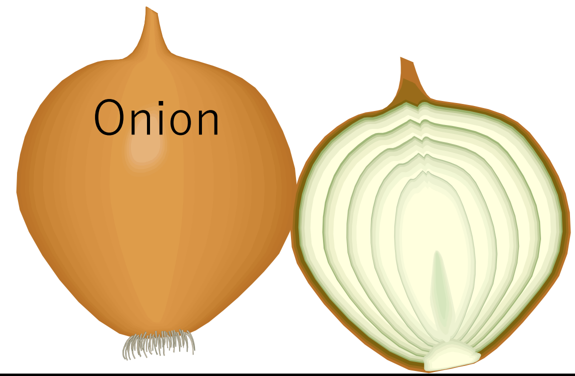
$$(x, y, z) \to (r, \theta, \varphi) \qquad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1}\left(z/\sqrt{x^2 + y^2 + z^2}\right) \\ \varphi = \tan^{-1}(y/x) \end{cases}$$



球座標系(Spherical Coordinates)







変数(Variables)	r,θ,φ
基底(Bases)	$\hat{r},\hat{\theta},\hat{\varphi}$
線素(Line element)	$d\boldsymbol{l}=\overrightarrow{PP'}=\hat{r}dr+\hat{\theta}rd\theta+\hat{\varphi}r\sin\theta\, d\varphi$
面素(Surface element)	$d\boldsymbol{S}_r=\hat{r}r^2\sin\theta\, d\theta d\varphi,\ d\boldsymbol{S}_\theta=\hat{\varphi}r\sin\theta\, drd\varphi,\ d\boldsymbol{S}_\varphi=\hat{z}rdrd\theta$
体積素(Volume element)	$dV=r^2\sin\theta\, drd\theta d\varphi$

球の体積

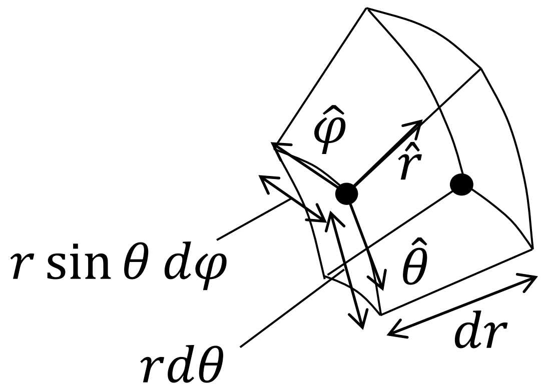
$$\iiint_V dV = \iiint_V r^2 \sin\theta \, dr d\theta d\varphi$$

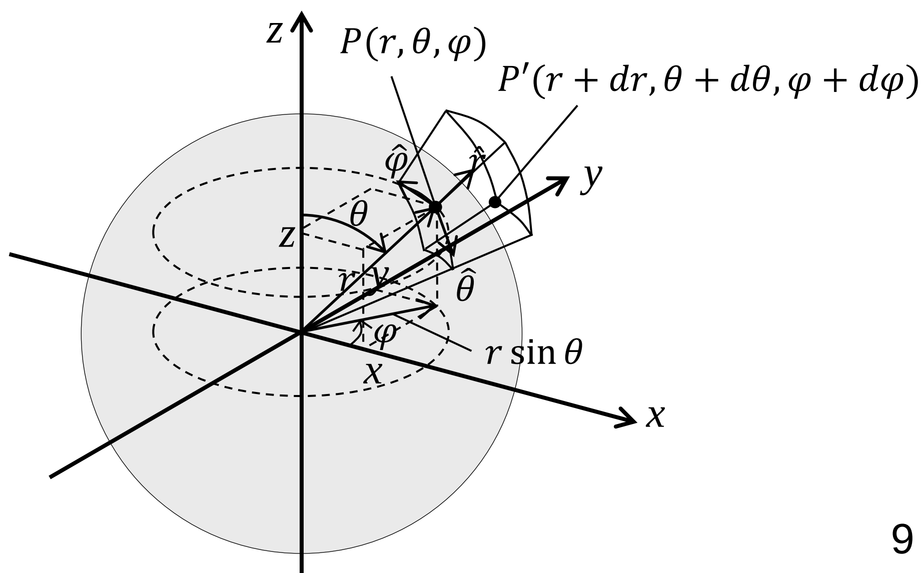
$$= \int_{r=0}^{r} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \int_{r=0}^{r} r^2 dr \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi$$

$$= \left[\frac{r^3}{3}\right]_0^r \cdot [-\cos\theta]_0^{\pi} \cdot [\varphi]_0^{2\pi}$$

$$= \frac{r^3}{3}\big(1-(-1)\big)\cdot 2\pi = \frac{4}{3}\pi r^3$$





体積積分のヤコビアン

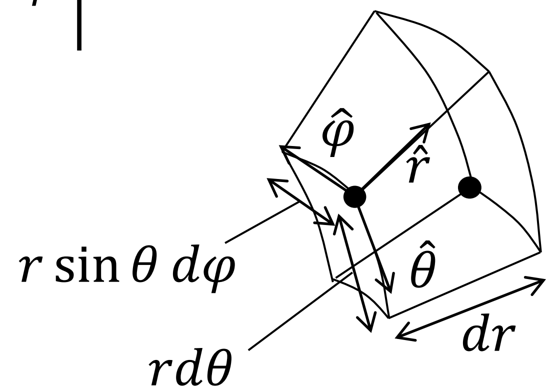
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

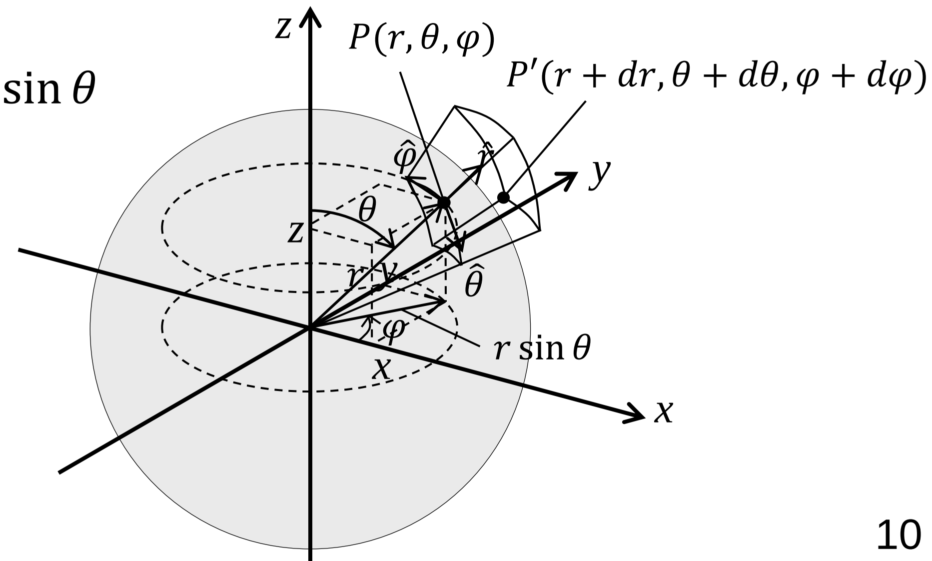
$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \left[\cos\theta \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix}\right]$$

$$= r^2[\cos\theta(\cos\theta\sin\theta) + \sin\theta\,(\sin^2\theta)] = r^2\sin\theta$$





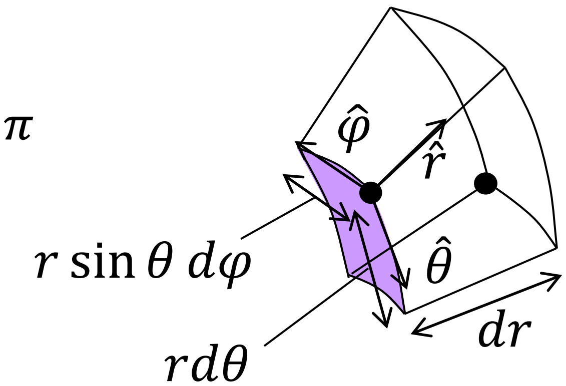
球の表面積

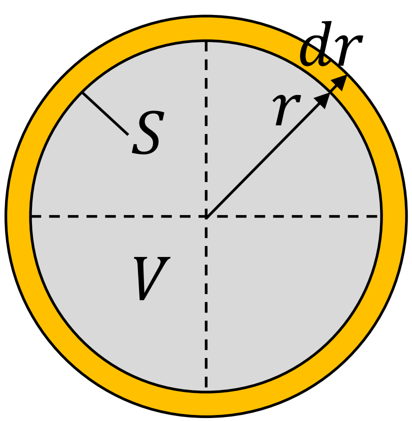
$$\iint_S \hat{r}\cdot d\boldsymbol{S}_r = \iint_S \hat{r}\cdot(\hat{r}r^2\sin\theta\, d\theta d\varphi)$$

$$= \iint_S r^2\sin\theta\, d\theta d\varphi = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^2\sin\theta\, d\theta d\varphi$$

$$= r^2\int_{\theta=0}^{\pi}\sin\theta\, d\theta\int_{\varphi=0}^{2\pi}d\varphi = r^2[-\cos\theta]_0^{\pi}\cdot[\varphi]_0^{2\pi}$$

$$= r^2\big(1-(-1)\big)\cdot 2\pi = 4\pi r^2$$





$$\frac{dV}{dr} = S$$

$$S = \frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$
$$= 4\pi r^2$$